International Journal of Theoretical Physics, Vol. 34, No. 8, 1995

States on Orthoalgebras

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Received March 28, 1995

The orthoalgebras, introduced by Foulis and Randall and studied by various authors, have recently become a significant mathematical structure of the logicoalgebraic foundation of quantum theories. In this paper we give a coherent account of states (=finitely additive measures) on orthoalgebras. In the first section we review basic properties of (and constructions with) orthoalgebras and develop a useful "pasting technique" (Theorem 1.12 and Proposition 1.16) applied later in this paper (and possibly elsewhere, too). We also exhibit orthoalgebras with rather interesting and "exotic" state spaces (Example 1.20 and Proposition 1.21). In the second section we construct orthoalgebras with preassigned state space properties. We prove a state representation theorem (Theorem 2.1) and obtain an orthoalgebraic version of Shultz's theorem (Theorem 2.7). In the third section we make a thorough analysis of the extension problem for states on orthoalgebras. We first study the orthoalgebras whose state spaces are finite dimensional. For these orthoalgebras we find a necessary and sufficient condition to allow extensions of states over larger orthoalgebras (Theorem 3.4). Then we prove that all Hilbertian orthoalgebras as well as all Boolean orthoalgebras allow extensions of states over larger orthoalgebras (Theorems 3.10 and 3.12).

INTRODUCTION

The logicoalgebraic foundation of quantum theories, which has been pursued quite intensely recently (e.g., Gudder, 1979; Kalmbach, 1983; Mączyński and Traczyk, 1973; Mittelstaedt, 1978; Pták and Pulmannová, 1991; Varadarajan, 1968) adopts for its underlying structure so-called "quantum logic." A quantum logic is usually assumed to be an orthomodular poset. Certain philosophical, physical, and mathematical considerations have suggested that the "right" logic of a quantum physical experiment is an orthoalgebra (OA). The OA is a partial algebra with a binary operation such that the orthomodular law can be derived as a consequence of the axioms imposed on

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the operation (see Definition 1.1. for the formal definition of the orthoalgebra). Viewed from the mathematical standpoint, the OAs are generalizations of orthomodular posets (and therefore they generalize orthomodular lattices and Boolean algebras).

There seem to be (at least) two explicit advantages of OAs over the quantum logics used previously. The first reason is physical—the OAs (unlike, e.g., orthomodular lattices) have come into existence on the grounds of plausible physical assumptions based on "test spaces" (e.g., Foulis and Randall, 1979, 1981). The second reason is technical—the axioms of an OA are natural and simple and the algebraic theory of OAs seems to meet well the needs encountered in applications (e.g., de Lucia and Dvurěcenskij, 1993; Foulis and Bennett, 1993; Foulis *et al.*, 1992; Rüttimann, 1989).

The main results to be proved in this paper have been stated in the abstract. As regards the technical aspects, it should be noted that the combinatorial and measure-theoretic constructions with OAs sometimes essentially differ from those used in orthomodular posets (see, for instance, the result formulated in Theorem 1.12). The proofs of combinatorially involved results are usually more transparent in OAs than in orthomodular posets. This allows us to obtain quite nontrivial state space representation results of Theorems 2.1 and 2.7 in an intuitive way (the reader may compare the proofs with in a sense analogous results in the orthomodular posets and lattices (see, e.g., Greechie, 1971; Navara *et al.*, 1988; Navara and Rogalewicz, 1991; Pták, 1987). On the other hand, some procedures with OAs enable a complete translation into orthomodular posets. For instance, Theorems 3.10 and 3.12, when specialized to orthomodular posets, considerably strengthen the results of Pták (1985). The OAs seem to be the most general "reasonable" orthomodular structures that allow the latter important extension theorems to be valid. (Finally, it should be observed that the usual investigation of two-valued and Jauch–Piron states is omitted here—the obvious reason is Proposition 2.18.)

1. BASIC CONSTRUCTIONS WITH ORTHOALGEBRAS. THE STATE SPACE OF AN ORTHOALGEBRA

Orthoalgebras are algebraic structures that generalize orthomodular posets. They were originally introduced in Randall and Foulis (1981). The following simplified definition, which we have adopted in this paper, is due to Golfin (1987).

Definition 1.1. An *orthoalgebra* (OA) is a quadruple $(L, 0, 1, \oplus)$ consisting of a set L containing two distinguished elements $0, 1 \in L$ and a partially defined binary operation $\oplus$ on L that satisfies the following conditions for all $a, b, c \in L$:

OA(i) [Commutativity law] If $a \oplus b$ is defined, then so is $b \oplus a$ and $a \oplus b = b \oplus a$.

OA(ii) [Associativity law] If $b \oplus c$ and $a \oplus (b \oplus c)$ are defined, then so are $a \oplus b$ and $(a \oplus b) \oplus c$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

OS(iii) [Orthocomplementation law] For each $a \in L$ there is a unique $b \in L$ such that $a \oplus b$ is defined and $a \oplus b = 1$.

OA(iv) [Consistency law] If $a \oplus a$ is defined, then $a = 0$.

The unique element b from OA(iii) is called the *orthocomplement* of a and denoted by a'. It can be easily seen that any OA L becomes an orthoposet if we define the ordering in L by putting $a \leq b \Leftrightarrow b = a \oplus c$ for a $c \in L$. Elements $a, b \in L$ are called *orthogonal* if $a \oplus b$ is defined. Let us now exhibit examples of OAs.

Example 1.2. Let L be a "quantum logic" [i.e., let L be an orthomodular poset; see, e.g., Gudder (1979) and Pták and Pulmannová (1991)]. If $a, b \in L$, let us define $a \oplus b$ exactly when a is orthogonal to b (i.e., when we have $a \leq b'$). In this case, $a \oplus b = a \vee b$. Then $(L, 0, 1, \oplus)$ is an OA.

If H is a Hilbert space, we denote by $L(H)$ the lattice of projections of H. In view of the above example, it is an orthoalgebra. Also, Boolean algebras may be regarded as examples of OAs. It was shown in Foulis *et al.* (1992) that an OA $(L, 0, 1, \oplus)$ arises as in the example above from a unique orthomodular poset if and only if the following condition is fulfilled in L: If $a, b, c \in L$ and $a \oplus b$, $a \oplus c$, and $b \oplus c$ are defined, then $a \oplus (b \oplus c)$ is defined, too. In the latter sense the orthoalgebras generalize orthomodular posets. If an OA cannot be induced by an orthomodular poset in the way determined in Example 1.2, we shall call it a *proper OA*. It should be noted that the above property characterizing proper OAs among orthomodular posets naturally appears in quantum event structures (Foulis and Randall, 1979; Randall and Foulis, 1981).

Example 1.3. Let L be the set of all idempotents of a ring with unity. Then L becomes an OA if $a \oplus b$ is defined exactly when $ab = ba = 0$, and in this case we put $a \oplus b = a + b$.

Indeed, basic ring-theoretic properties yield that the partial operation $\oplus$ defined in Example 1.3 satisfies the conditions OA(i)–OA(iv) (see also Foulis and Bennett, 1993). It should be noted that there are OAs that do not arise this way (Navara, 1994).

Definition 1.4. Let K, L be OAs. A mapping $f: K \to L$ is called an OA morphism if $f(1) = 1$ and if the following condition is satisfied: If $a \oplus b$ is defined in K, then $f(a) \oplus f(b)$ is defined in L and $f(a \oplus b) = f(a) \oplus f(b)$. An OA morphism $f: K \to L$ is called an *embedding* if f is injective and, moreover, for any triple $a, b, c \in K$ the following condition is fulfilled: The equality $a \oplus b = c$ is defined and valid in K if and only if the equality $f(a) \oplus f(b) = f(c)$ is defined and valid in L. An embedding f: $K \to L$ is called an *isomorphism* if f is surjective (in this case K and L are called isomorphic).

Obviously, if f: $K \to L$ is an embedding, then we can understand the set $f(K)$ (with the operation inherited from L) as an OA in its own right. Thus, we may (and sometimes shall) identify K with a subset of L and call K a *suborthoalgebra* of L (sub-OA of L). Dually, if the latter case occurs, we may also call L an *enlargement* of K.

Example 1.5. Let $L = \{0, 1, a, b, c, d, e, f, a', b', c', d', e', f'\}$ be a set. Let us define, apart from obvious relations,

$$a' = c \oplus f = d \oplus e, \qquad c' = a \oplus f = b \oplus e, \qquad e' = a \oplus d = b \oplus c$$

Observe that

$$b' = c \oplus e, \qquad d' = a \oplus e, \qquad f' = a \oplus c$$

Then L becomes an OA (see Fig. 1 for a schematic depiction of L). One easily sees that L is not an orthomodular poset. Indeed, $a \oplus c$, $a \oplus e$, and $c \oplus e$ are defined, but $a \oplus (c \oplus e)$ is not (Foulis *et al.*, 1992). Let us call this (proper) OA the *triangle OA* (the Wright triangle). The latter OA will be often used in the constructions that follow. Observe that the triangle OA possesses three three-atomic Boolean sub-OAs (they are indicated in Fig. 1 by the sides of the triangle).

Example 1.6. Let $L = \{0, 1, a, b, c, d, e, f, g, a', b', c', d', e', f', g'\}$. Let the elements of $L \backslash \{g, g'\}$ be subject to the same requirements as in Example 1.5, and let us add the following relations: $g' = c \oplus d = e \oplus f$, $c' = d \oplus g$, $e' = f \oplus g$, $b' = d \oplus f$, $d' = b \oplus f = c \oplus g$, and $f' = b \oplus$

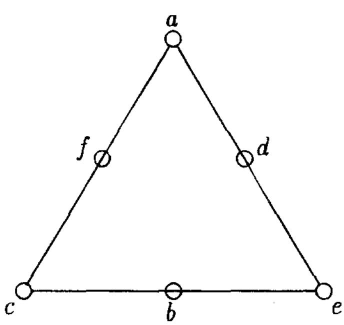

Fig. 1.

$d = e \oplus g$. Then L becomes an OA. (See Fig. 2 for a schematic depiction.) Let us call this (proper) OA the *triangle-circle OA*. Observe that the triangle-circle OA has six three-atomic Boolean sub-OAs (they are indicated in Fig. 2 as the sides of the triangle, two side-bisectors, and the circle). Moreover, the triangle OA is a sub-OA of the triangle-circle OA.

Proposition 1.7. An OA L is proper if and only if it contains the triangle OA as a sub-OA in such a way that $a \oplus (c \oplus e)$ is not defined in L (we use the notation of Example 1.5).

Proof. According to Foulis *et al.* (1992), an OA L is proper if there are $a, c, e \in L$ such that $a \oplus c$, $a \oplus e$, $c \oplus e$ are defined in L, but $a \oplus (c \oplus e)$ is not. If such elements exist in L, then they generate the triangle OA. Indeed, the nonexistence of $a \oplus (c \oplus e)$ implies that a, c, e are nonzero. Suppose, for instance, that $(c \oplus e)' = 0$. Then there are elements $d, f \in L$ such that $e = c' = a \oplus f$ and $c = e' = a \oplus d$. Since $c \oplus e = (a \oplus f) \oplus (a \oplus d) = (a \oplus a) \oplus (f \oplus d)$ is defined, we infer that $a = 0$. This is a contradiction. We argue similarly in the cases $(a \oplus c)' = 0$ or $(a \oplus e)' = 0$. ■

Remark 1.8. It should be noted that the second condition in Proposition 1.7 is not superfluous. Indeed, the OA depicted by its Greechie diagram in Fig. 3 is an orthomodular poset (Dichtl, 1981), which contains the triangle OA as a sub-OA. However, the term $a \oplus (c \oplus e)$ is defined there.

Let us now take up basic constructions with OAs. They will be frequently used later.

Definition 1.9. Let $\{L_\alpha | \alpha \in I\}$ be a collection of OAs. Let us denote by L the Cartesian product of L_α's (thus, $L = \Pi_{\alpha \in I} L_\alpha$). Let us further endow L with 0, 1 and with $\oplus$ coordinatewise. Then L is called a *product* of $\{L_\alpha | \alpha \in I\}$.

One can easily check that a product of OAs becomes again an OA and any projection $\pi_\alpha: L \to L_\alpha$ becomes an OA morphism onto L_α ($\alpha \in I$).

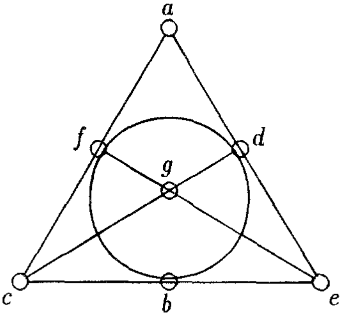

Fig. 2.

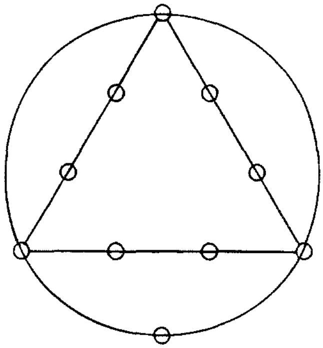

Fig. 3.

Let us now define the notion of an interval in an OA. Suppose that $(L, 0, 1, \oplus)$ is an OA and suppose that $a \in L$, $a \neq 0$. Consider the set $[0, a]_L = \{b \in L \mid b \leq a\}$ endowed with a partial operation $\oplus_a$ defined as follows: $\oplus_a = \oplus \cap [0, a]_L^3$ (i.e., $b \oplus_a c = d$ if and only if $b, c, d \in [0, a]_L$ and $b \oplus c = d$). Then $([0, a]_L, 0, a, \oplus_a)$ is called an *interval* in L.

Proposition 1.10. The interval $([0, a]_L, 0, a, \oplus_a)$ is an OA.

Proof. The conditions OA(i) and OA(iv) are fulfilled automatically. As for OA(iii), we see that if $b \in [0, a]_L$, then $a = b \oplus c$ for a unique element c. As for OA(ii), if $(b \oplus_a c) \oplus_a d$ is defined in $[0, a]_L$, then $a = ((b \oplus_a c) \oplus_a d) \oplus e = ((b \oplus c) \oplus d) \oplus e = (b \oplus (c \oplus d)) \oplus e = (b \oplus_a (c \oplus_a d)) \oplus e$. Since the cancellation law is valid in OAs, which is a direct consequence of the axioms of an OA (see also Foulis *et al.*, 1992; Rüttimann, 1989), we see that $(b \oplus_a c) \oplus_a d = b \oplus_a (c \oplus_a d)$. ∎

We shall now describe a "pasting" technique for OAs which enables us to construct OAs with preassigned intrinsic or state-space properties. Prior to that, let us agree to call $b \in L$ a *relative complement* of c in $[0, a]_L$ if $c \oplus b = a$. We shall sometimes use the notation $b = a \ominus c$.

Definition 1.11. Let $\mathscr{A}$ be a family of OAs that satisfies the following conditions for all distinct K, L in $\mathscr{A}$:

(P1) $1_K = 1_L$ (this means that all greatest elements coincide).

(P2) If $a, c \in K \cap L$ and if $a \in [0, c]_K \cap [0, c]_L$, then the relative complement of a in $[0, c]_K$ agrees with the relative complement of a in $[0, c]_L$.

(P3) If $a, d \in K$, $b \in L$, and $c \in K \cap L$ and if $a \oplus_K d = c$ and $c \oplus_L b$ is defined, then there is an $M \in \mathscr{A}$ such that $a \oplus_M d = c$ and $c \oplus_M b$ is defined.

Further, put $P = \cup_{L \in \mathcal{A}} L$ and define a partial operation $\oplus$ on P by setting $\oplus = \cap_{L \in \mathcal{A}} \oplus_L$ (that is, we set $a \oplus b = c$ if and only if there is an $L \in \mathcal{A}$ such that $a \oplus_L b = c$). Also, define $1 = 1_L$, $0 = 0_L$, where L is an arbitrary element of $\mathcal{A}$. Let us call the resulting quadruple $(P, 0, 1, \oplus)$ the *pasting* of $\mathcal{A}$.

Theorem 1.12. If a collection $\mathcal{A}$ of OAs satisfies the conditions (P1)–(P3), then its pasting, $(P, 0, 1, \oplus)$, is an OA.

Proof. The quadruple $(P, 0, 1, \oplus)$ obviously fulfills OA(i) and OA(iv) as a direct consequence of (P1) and (P2). The condition OA(iii) follows from (P1) and (P2), too. The condition OA(ii) is ensured by (P3). ∎

Remark 1.13. For later use, let us observe that the condition (P3) in the above construction is weaker than the following condition:

(P3+) If $d \in K \cap L$, then either $[0, d]_K = [0, d]_L$ or $[0, d']_K = [0, d']_L$.

Observe also that if the collection $\mathcal{A}$ satisfies the condition (P3+), the verification of (P2) is considerably simpler. We shall frequently consider pastings that satisfy this stronger condition (P3+).

The pasting technique described above yields the following results. [Recall first that a maximal Boolean sub-OA in an OA L is called a *block* of L. One can easily prove by the standard use of Zorn's lemma that every Boolean sub-OA of L can be extended to a block in L. Taking advantage of this observation, one immediately sees that the collection of all blocks in an OA fulfills the conditions (P1)–(P3).]

Theorem 1.14. Every OA can be regarded as the pasting of the collection $\mathcal{A}$ of all its blocks.

A transparent way of depicting a finite OA and its block configuration is the so-called *Greechie diagram* [see Greechie (1971) for the original use of this idea]. The Greechie diagram of an OA L is a hypergraph whose vertices correspond to atoms of L and edges correspond to blocks of L. For instance, Fig. 1 (resp. Fig. 2) is the Greechie diagram of the triangle OA (resp. the triangle-circle OA).

The pasting constructions that follow will be used for the state space investigations in the next paragraphs. Let K, L be disjoint OAs and let $a \in K$ be an atom in K. (An element $a \in K$ is called an *atom* in K if the inequalities $0 < b \leq a$ imply $a = b$.) Let us consider the OA product $[0, a']_K \times L$. For each $d \in [0, a']_K$, let us define $h(d) = (d, 0_L) \in M$ and $h(d') = (d', 1_L) \in M$. Obviously, h is an isomorphism between the sub-OA $[0, a']_K \cup [a, 1]_K$ of K and the sub-OA $\{(d, e) \in M \mid d \in [0, a']_K, e \in \{0_L, 1_L\}\}$ of M.

Let us now consider such isomorphic copies of K and M—denoted again by K and M—that $d \in K$ equals $h(d) \in M$ for all $d \in [0, a']_K \cup [a, 1]_K$. Upon the foregoing identification, we obtain the following result.

Proposition 1.15. Let K, L be OAs and let $a \in K$ be an atom in K. If we put $M = [0, a'] \times L$ and understand K, M subject to the identification described above, then K, M fulfil the conditions (P1), (P2), and (P3+) of the pasting. Moreover, if P is the pasting of K and M, then $[0, a]_P$ is OA-isomorphic to L.

Proof. The condition (P1) is satisfied trivially. The condition (P3+) is fulfilled, too, for if $d \in K \cap M$, then either $[0, d]_K = [0, d]_M$ (provided $d \in [0, a']_K$) or $[0, d']_K = [0, d']_M$ (provided $d' \in [0, a']_K$). The property (P2) follows from h being an isomorphism. ■

The intuitive content of the latter construction suggests the following convention. We shall say that the OA P of Proposition 1.15 is formed by *substituting the atom $a \in K$ by the orthoalgebra L.*

The latter construction can be further generalized the following way. Suppose that K, L are orthoalgebras. Suppose further that b is an element of $L \setminus \{0, 1\}$ and a is an atom in K. Let M denote the OA we obtain by substituting the atom $a \in K$ by the orthoalgebra $[0, b]_L$. Take now such isomorphic copies of L and M—denoted again by L and M—that the corresponding elements of $[0, b]_L$ and $[0, a]_M$ are identified and so are also their complements. It is easy to see that the latter OAs L and M fulfil the conditions (P1), (P2), and (P3+) of the pasting. Let P denote the pasting of L and M. We then say that P is formed by *connecting the atom $a \in K$ with the element $b \in L$.* (It should be noted that $L \cap M = [0, b]_L \cup [b', 1]_L$.)

The latter construction can be performed simultaneously for a collection of orthoalgebras provided the elements b_α $(\alpha \in I)$ of the "fixed" OA L are in some sense independent. The following proposition gives a precise formulation of this result.

Proposition 1.16. Let L and K_α $(\alpha \in I)$ be OAs. For any $\alpha \in I$, let a_α be an atom of K_α and b_α be an element of $L \setminus \{0, 1\}$. Let P_α be the OA which is formed by connecting $a_\alpha \in K_\alpha$ with $b_\alpha \in L$. If for any $\alpha, \beta \in I$ the partial operation $\oplus$ is not defined for the couple b'_α, b'_β, then the collection $\{P_\alpha \mid \alpha \in I\}$ fulfills the conditions (P1), (P2), and (P3+) of the pasting. A corollary: The collection $\{P_\alpha \mid \alpha \in I\}$ can be pasted in the sense of Proposition 1.12 to give rise to its pasting, P.

Proof. By the construction of P_α $(\alpha \in I)$, we immediately see that $\{P_\alpha \mid \alpha \in I\}$ admits a pasting as soon as we verify that $\{P_\alpha \mid \alpha \in I\}$ fulfills the condition (P3+) (see Remark 1.13). First, we know that $P_\alpha \cap L = [0, b_\alpha]_L \cup [b'_\alpha, 1]_L$. Consider now the set $P_\alpha \cap P_\beta$ $(\alpha \neq \beta)$. Obviously, $P_\alpha \cap P_\beta$ is a sub-OA of L. Choose an element $c \in P_\alpha \cap P_\beta$. If $c \in [0, b_\alpha]_L \cap [0, b_\beta]_L$, then $[0, c]_{P_\alpha} = [0, c]_{P_\beta} = [0, c]_L \subset P_\alpha \cap P_\beta$. If $c \in [b'_\alpha, 1]_L \cap [b'_\beta, 1]_L$, then $[c, 1]_{P_\alpha} = [c, 1]_{P_\beta} \subset P_\alpha \cap P_\beta$. Finally, suppose that $c \in [0, b_\beta]_L \cap$

$[b'_\alpha, 1]_L$. This implies that $b'_\alpha \leq b_\beta$ and therefore the operation $b'_\alpha \oplus b'_\beta$ is defined—a contradiction. The proof is complete. ■

Under the assumptions of the latter proposition, we say that the OA P is obtained by *connecting the atoms* $a_\alpha \in K_\alpha$ *with* $b_\alpha \in L$ ($\alpha \in I$).

Let us now take up the states on OAs.

Definition 1.17. Let L be an OA. A mapping s: $L \to [0, 1]$ is called a *state* on L if:

(i) $s(1) = 1$.
(ii) $s(a \oplus b) = s(a) + s(b)$ provided $a, b \in L$ and $a \oplus b$ is defined in L.

The notion of state on L, which may formally be regarded as a generalization of the notion of probability measure, can be associated with the physical "state" if we understand L as a "quantum logic" (see, e.g., Gudder, 1979; Pták and Pulmannová, 1991; Varadarajan, 1968). In this paper we analyze the state space of an orthoalgebra purely mathematically (though the motivation for our consideration often comes from quantum physics). Let us denote by $\mathcal{S}(L)$ the set of all states on L.

The following results illustrate basic properties of the states on an OA and set the stage for the investigation in the following sections. Let us first observe that a proper OA cannot be too near to either a Boolean algebra or a projection lattice.

Proposition 1.18. (i) Let L fulfil the following property: If $a, b \in L$ and $a \oplus b'$ is not defined, then there exists a state $s \in \mathcal{S}(L)$ such that $s(a) = 1$ and $s(b) < 1$. Then L is not proper (i.e., L is an orthomodular poset).

(ii) Let L fulfil the following property: If $a \in L$ and $a \neq 0$, then there is a Jauch–Piron state $s \in \mathcal{S}(L)$ with $s(a) = 1$. [A state $s \in \mathcal{S}(L)$ is called *Jauch–Piron* (Rüttimann, 1977) if the equality $s(c) = s(d) = 1$ for $c, d \in L$ implies the existence of an element $e \in L$ such that $e \leq c$, $e \leq d$, and $s(e) = 1$.] Then L is not proper. A corollary: If L is finite and all states on L are Jauch–Piron, then L is a Boolean algebra.

Proof. By Proposition 1.7, if L is proper, then it contains the triangle OA as its sub-OA and $a \oplus (c \oplus e) = a \oplus b'$ is not defined in L (here we have preserved the notation of Example 1.5). If $s \in \mathcal{S}(L)$ and $s(a) = 1$, then $s(c) = s(e) = 0$ and $s(b) = 1$. Thus, the state s neither fulfills the condition of Proposition 1.18 (i) nor it is Jauch–Piron. [The corollary follows then from Bunce *et al.* (1985).]

The next proposition shows that states on a finite product of OAs are generated by the coordinate states.

Proposition 1.19. Let $\{L_i | i \leq n\}$ be a finite collection of OAs and let $L = \Pi_{i \leq n} L_i$ be the OA product of $\{L_i | i \leq n\}$. Let $s_i \in \mathcal{S}(L_i)$ $(i \leq n)$ and let $\{\alpha_i | i \leq n\}$ be such a set of nonnegative numbers that $\Sigma_{i \leq n} \alpha_i = 1$. For any i $(i \leq n)$, let π_i: $L \to L_i$ be the canonical projection onto L_i and let us define a state on L, some $\tilde{s}_i$, by putting $\tilde{s}_i = s_i \circ \pi_i$. Then the mapping $s = \Sigma_{i \leq n} \alpha_i \tilde{s}_i$ is a state on L. Moreover, all states on L can be represented the latter way.

The proof is straightforward (see also Maňasová and Pták, 1981).

As an application of Proposition 1.19 (and as a preliminary to the next section) let us show that every finite-dimensional simplex can be viewed as a state space of a finite proper OA. We begin with the following example.

Example 1.20. Let L be the (proper) OA depicted by its Greechie diagram in Fig. 4 (the *kite* OA). Then L does not possess any state [thus, $\mathcal{S}(L) = \emptyset$].

Proof. It is easy to check that the configuration of the blocks in Fig. 4 satisfies the conditions (P1), (P2), and (P3+) for the pasting. Let us now assume that $s \in \mathcal{S}(L)$ and look for a contradiction. We first have

$$[s(a) + s(b) + s(c)] + [s(d) + s(e) + s(f) + s(g)] \\ + [s(h) + s(k) + s(m)] = 1 + 1 + 1 = 3$$

(the terms in the brackets are sums over blocks in L). Moreover,

$$[s(a) + s(d) + s(k)] + [s(b) + s(f) + s(m)] + [s(c) + s(e) + s(h)] = 3$$

and therefore $s(g) = 0$. Similarly,

$$[s(a) + s(g) + s(m)] + [s(b) + s(d) + s(h)] + [s(c) + s(f) + s(k)] = 3$$

and therefore $s(e) = 0$. Finally,

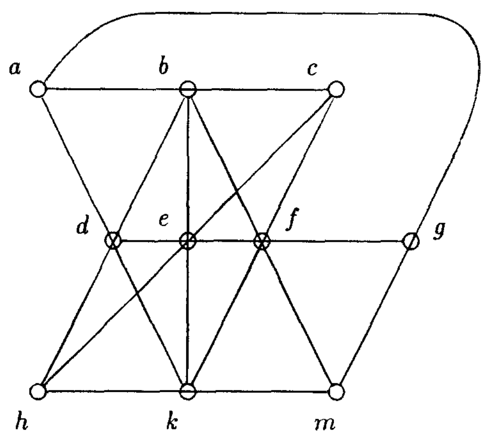

Fig. 4.

$$[s(b) + s(e) + s(k)] + [s(c) + s(e) + s(h)] + [s(a) + s(g) + s(m)] = 3$$

and therefore $s(d) + s(f) = 0$. We infer that $s(d) + s(e) + s(f) + s(g) = 0$, which is a contradiction. This proves that $\mathcal{S}(L) = \emptyset$. ∎

Proposition 1.21. Suppose that L is a (finite) stateless OA [$\mathcal{S}(L) = \emptyset$]. Then there is a (finite) proper OA K such that card $\mathcal{S}(K) = 1$ and such that there is an epimorphism f: $K \to L$. Moreover, the unique state of $\mathcal{S}(K)$ is two-valued.

Proof. If L is a stateless OA and if we put $K = L \times \{0, 1\}$, where $\{0, 1\}$ is a two-element Boolean algebra, then card $\mathcal{S}(K) = 1$ (Proposition 1.19). Indeed, the unique state $s \in \mathcal{S}(K)$ is the two-valued state defined by the formula $s((k, i)) = 1$ if and only if $i = 1$. Obviously, the projection π: $K \to L$ is an epimorphism. ∎

The result formulated below Proposition 1.19 can be now seen easily. If L is such a finite OA that $\mathcal{S}(L) = 1$ and if $L_i = L$ for any $i \leq n$, then $\mathcal{S}(\Pi_{i \leq n} L_i)$ is an n-dimensional simplex (Proposition 1.19). The next section gives a considerable deepening of the latter result.

2. ORTHOALGEBRAS WITH PREASSIGNED STATE SPACES

It can be shown easily that for any OA L the state space $\mathcal{S}(L)$ of L is a convex set. Moreover, this set becomes a compact topological space when we view it as a subspace of R^L, where R^L is supposed to be endowed with the "pointwise" topology [in fact, $\mathcal{S}(L) \subset [0, 1]^L$]. In what follows we shall prove that the "vice versa" statement is also true: Every compact convex set in R^I (for a set I) can be represented as a state space of a proper orthoalgebra [the representation means, as usual, the existence of an affine homeomorphism between $\mathcal{S}(L)$ and the given compact convex set]. But we shall show that a considerably stronger result is true: The latter OA L may be required of an "arbitrary degree of noncompatibility" (i.e., L may be required to contain an arbitrary preassigned OA as its sub-OA). This property—besides being interesting in its mathematical content—seems of importance as far as a potential application in quantum theories is concerned.

Theorem 2.1. Let L be an OA and let S be a compact convex subset of the state space $\mathcal{S}(L)$. Then there is an OA K with the following properties:

(i) L is a sub-OA of K.

(ii) A state $s \in \mathcal{S}(L)$ admits an extension over K if and only if $s \in S$; moreover, if the extension of s exists, then it is unique. A corollary: S is affinely homeomorphic to $\mathcal{S}(K)$.

(iii) If s belongs to the interior of S, Int S, and if s is strictly positive [i.e., $s(a) > 0$ whenever $a \neq 0$], then the (unique) extension of s over K is also strictly positive.

Proof. We shall need a few auxiliary propositions.

Proposition 2.2. Let L be the triangle-circle OA (see Example 1.6) and let s be a state on L. Then $s(a) = s(b) = s(g)$ and $s(c) = s(d) = s(e) = s(f) = (1 - s(a))/2$. A corollary: If $s(a) \in (0, 1)$ for $s \in \mathcal{S}(L)$, then s is strictly positive.

Proof. Considering the corresponding Greechie diagram (see Fig. 2), we consecutively obtain

$$2s(a) = 1 - s(c) - s(f) + 1 - s(d) - s(e) = 2s(b) = 2s(g)$$

Thus, $s(a) = s(b) = s(g)$ and $s(c) = s(d) = s(e) = s(f) = (1 - s(a))/2$. ∎

Proposition 2.3. Suppose that we are given a natural number $n \in N$, $n \geq 2$. Then there is a finite proper OA X_n which possesses two atoms $c, d \in X_n$ with the following properties:

(i) If s is a state on X_n, then $s(d) = (1/n)(1 - s(c))$.

(ii) For each $\alpha \in [0, 1]$ there is a unique state $s \in \mathcal{S}(X_n)$ such that $s(c) = \alpha$ [in particular, every state $s \in \mathcal{S}(X_n)$ is uniquely determined by its value at c]. Moreover, if $\alpha \in (0, 1)$, then the latter state $s \in \mathcal{S}(X_n)$ is strictly positive.

Proof. If n is even ($n = 2k$), then we can take for X_n the OA determined by its Greechie diagram in Fig. 5 (the *tree* OA). Since X_n is built up of copies of the triangle-circle OA and since the configuration of blocks is legitimate for the pasting, Proposition 2.3 follows immediately from Proposition 2.2 and Theorem 1.12. If n is odd ($n = 2k + 1$), then it suffices to add an atom d_{2k+1} to the "long" vertical block (see the dashed line in Fig. 5). ∎

Prior to the next result, let us recall the notion of an M-base in OAs [see Marlow (1978) for the original use of this notion]. Let L be an OA and let M be a subset of L. Then M is called an *M-base* in L if M satisfies the following two conditions:

(i) If $a, b \in M$ and a, b are distinct, then $a \oplus b$ is not defined in L.
(ii) If $a \in L$, then either $a \in M$ or $a' \in M$.

It can be proved by a standard Zorn's lemma argument that there exists an M-base in any OA.

Proposition 2.4. Let M be an M-base in L. Then the following statements hold true:

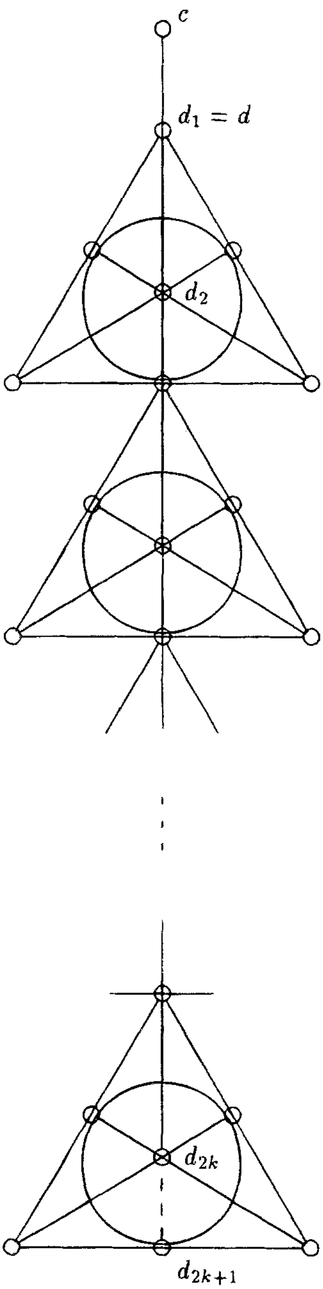

Fig. 5.

(i) If $s, t \in \mathcal{S}(L)$ and $s(a) = t(a)$ for any $a \in M\setminus\{1\}$, then $s = t$.

(ii) If we are given orthoalgebras L_a with atoms $b_a \in L_a$ $(a \in M\setminus\{1\})$, then we can construct the pasting P formed by connecting each atom $b_a \in L_a$ with $a' \in L$.

Proof. Part (i) is obvious and part (ii) immediately follows from Proposition 1.16. ■

Proposition 2.5. Let L be an OA and let p, p_i $(i \leq k)$ be rational numbers. Let $a_i \in L$ $(i \leq k)$ and let T be the set of all states on L that satisfy the following inequality:

$$\text{(H)} \qquad \sum_{i\leq k} p_i s(a_i) + p \geq 0$$

Then there is an OA K such that the following three conditions are satisfied:

(i) L is a sub-OA of K.

(ii) A state $s \in \mathcal{S}(L)$ has an extension over K if and only if $s \in T$; moreover, if the extension of s over K exists, then it is unique.

(iii) If $s \in \mathcal{S}(L)$ is strictly positive and if s satisfies a strict inequality of (H), then the extension of s over K is also strictly positive.

Proof. Let M be an M-base in L fixed from now on. Put $M' = \{b \in L \mid b = a' \text{ for } a \in M\}$. If $a_i \notin M'$, then $a_i \in M$ and we may substitute $1 - s(a_i')$ for $s(a_i)$ into (H) preserving the type of the equality (the coefficients p, p_i will change, of course). It follows that without any loss of generality we may (and shall) suppose that $a_i \in M'$ for all i $(i \leq k)$. Omitting the zero summands, we shall suppose that $p_i \neq 0$ and $a_i \neq 0$ $(i \leq k)$. Thus, Proposition 2.4(ii) is applicable to a_i $(i \leq k)$.

We are now going to discuss in turn the possibilities which may occur. Suppose first that there is an i $(i \leq k)$ such that $p_i < 0$. We then claim that we can construct an enlargement of L, some $\tilde{L}$, such that all states on L admit a unique extension over $\tilde{L}$ and, moreover, the original condition (H) transforms into a condition of the form

$$\sum_{j\leq k, j\neq i} p_j s(a_j) - p_i s(\tilde{a}_i) + \tilde{p} \geq 0$$

Thus, by induction, we shall be able to assume that all coefficients p_i $(i \leq k)$ are positive.

Let us prove the statement we claimed above. Let us take the triangle-circle OA, defined in Example 1.6 and denoted here by Q, and let us connect the distinguished atom $a \in Q$ (in the notation of Fig. 2) with $a_i \in L$. Let us denote the resulting OA by $\tilde{L}$. By the properties of Q (see Proposition 2.2), each state on L admits a unique extension to a state on $\tilde{L}$. If we now consider the element $c \oplus e$ of Q (in the notation of Fig. 2) and if we understand it as an element of $\tilde{L}$ (this may be assumed since Q is a sub-OA of $\tilde{L}$), we see that $s(a_i) = 1 - s(c \oplus e)$ for any state $s \in \mathcal{S}(\tilde{L})$. We therefore obtain $p_i s(a_i) = p_i + (-p_i)s(c \oplus e)$ and we may put $\tilde{a}_i = c \oplus e$ and $\tilde{p} = p + p_i$ [the M-base M on L can be easily extended to an M-base on $\tilde{L}$ that contains $(c \oplus e)'$]. This proves the required statement.

It follows that without any loss of generality we may assume that all the coefficients $p_i (i \leq k)$ are positive. Let us rewrite the condition (H) in the form

$$(\overline{\mathrm{H}}) \qquad \sum_{i \leq k} p_i(1 - s(a_i)) \leq q$$

where $q = p + \Sigma_{i \leq k}\, p_i$. The latter condition $(\overline{\mathrm{H}})$ is obviously equivalent to the condition (H) and, moreover, the left-hand side of $(\overline{\mathrm{H}})$ is always nonnegative. Let us now discuss the alternatives for the values of q.

First, suppose that $q < 0$. Then we are to embed the original OA L into a stateless OA. Take the kite OA (see Example 1.20) and denote it by X. If we now connect an arbitrarily chosen atom of X with an arbitrary element of $L \setminus \{0, 1\}$, then we obtain the required stateless OA, K, that contains L as its sub-OA.

Second, suppose that $q = 0$. Then our condition $(\overline{\mathrm{H}})$ is equivalent to the requirement of $s(a_i) = 1$ for all $i \leq k$. Let us choose an OA, say V, such that card $\mathcal{S}(V) = 1$ and the unique $s \in \mathcal{S}(V)$ is two-valued (such OAs do exist; see Proposition 1.21). Take OA-isomorphic copies of V, some V_i $(i \leq k)$, and pick atoms $c_i \in V_i$ such that $s_i(c_i) = 1$ for any $s_i \in \mathcal{S}(V_i)$ $(i \leq k)$. Let us now connect the elements $c_i \in V_i$ with $a_i \in L$ (this is legitimate since $a_i \in M' \setminus \{0\}$ for the M-base M—see Proposition 2.4(ii)). It is obvious that the OA K arising by this pasting enlarges L and satisfies $s(a_i) = 1$ $(i \leq k)$ for any $s \in \mathcal{S}(K)$.

Third, suppose that $q > 0$. Then we can rewrite the condition $(\overline{\mathrm{H}})$ in the form

$$\sum_{i \leq k} \frac{p_i}{q}(1 - s(a_i)) \leq 1$$

Since q, p_i $(i \leq k)$ are rational numbers, we can by repeating a_i rewrite the above inequality in the form

$$(\overline{\overline{\mathrm{H}}}) \qquad \sum_{j \leq m} \frac{1}{n_j}(1 - s(\tilde{a}_j)) \leq 1$$

where n_j $(j \leq m)$ are natural numbers strictly greater than 1 and $\{\tilde{a}_j | j \leq m\} = \{a_i | i \leq k\}$. For all j $(j \leq m)$, let us now take a copy of the tree OA X_{n_j} (Fig. 5) with atoms c_j, resp. b_j, corresponding to the atoms c, resp. d, in Fig.

5. For all $j \leq m$, connect c_j with $\tilde{a}_j \in L$. What we obtain as the result of the pasting is an OA $\tilde{K}$. Considering now the elements b_j as elements of $\tilde{K}$, we see (Proposition 2.3) that every state $s \in L$ admits a unique extension to a state t on $\tilde{K}$ and, moreover,

$$t(b_j) = \frac{1}{n_j}(1 - s(\tilde{a}_j)) \qquad (j \leq m)$$

We will now enlarge the OA $\tilde{K}$ by adding a block that contains all of the elements $b_j \in \tilde{K}$ $(j \leq m)$. To this end, let B be the Boolean algebra with $m + 1$ atoms $b_1, \ldots b_m, e$, where e is a new atom. Let us now take for K the pasting of $\tilde{K}$ and B. One easily sees that if $t \in \mathcal{S}(\tilde{K})$ is the extension of $s \in \mathcal{S}(L)$, then t admits an extension over K exactly if $\Sigma_{j \leq m} t(b_j) \leq 1$. This equality is equivalent to (H) and the proof of Proposition 2.5 is complete. ∎

Proposition 2.6. Let ω be an ordinal number. Let $\{L_\alpha \mid \alpha < \omega\}$ be a collection of OAs. For any pair α, β with $\alpha \leq \beta < \omega$ let there be an OA-embedding $e_{\beta,\alpha}$: $L_\alpha \to L_\beta$ such that $e_{\alpha,\alpha} = id_{L_\alpha}$ and, moreover, $e_{\gamma,\alpha} = e_{\gamma,\beta} \circ e_{\beta,\alpha}$ for any triple α, β, γ with $\alpha \leq \beta \leq \gamma < \omega$. Then there is an *inductive limit* of $\{L_\alpha \mid \alpha < \omega\}$ in OAs. In other words, there is an OA L and a collection f_α: $L_\alpha \to L$ $(\alpha \in W_\omega)$ of OA-embeddings such that the following two conditions are satisfied:

(i) $L = \cup_{\alpha<\omega} f_\alpha(L_\alpha)$.
(ii) If $\alpha \leq \beta < \omega$, then $f_\beta = e_{\beta,\alpha} \circ f_\alpha$.

Proof. Put $W = \cup_{\alpha<\omega} (\Pi_{\alpha \leq \beta < \omega} L_\beta)$ and consider the subset Y of W determined as follows: An element $w = (w_\alpha, w_{\alpha+1}, \ldots) \in W$ belongs to Y if and only if $w_\beta = e_{\beta,\alpha}(w_\alpha)$ for any β, $\alpha \leq \beta < \omega$. Let us now factorize the set Y via such an equivalence θ that we have $w \;\theta\; v$ exactly when there is a $\gamma < \omega$ with $w_\gamma = v_\gamma$. Let us denote the set Y/θ by L and convert L into an OA by canonical copying of 0, 1, and $\oplus$ from W into L. It can be seen easily that L is the desired limit. ∎

Let us now return to the proof of Theorem 2.1. Since the given S $[S \subset \mathcal{S}(L)]$ can be viewed as a compact convex subset of the (locally convex topological linear space) R^L, it is well known from the convexity theory (see, e.g., Schaefer, 1971) that S is the intersection of all closed half-spaces in R^L that contain S. Every closed half-space is determined by a hyperplane. Taking into account the description of the hyperplanes in the "weak" topological liner space R^L and approximating every hyperplane by hyperplanes with rational coefficients, we finally see that S can be determined by a collection $\mathcal{H}$ of inequalities of the type (H) (the M-base in question being taken arbitrarily but uniformly for all inequalities in $\mathcal{H}$). We have to prove that for any collection $\mathcal{H}$ of inequalities there is an enlargement of L, some K, such that a state $s \in \mathcal{S}(L)$ admits a unique extension over K if and only if s fulfills all the inequalities in $\mathcal{H}$.

Let us well-order the set $\mathcal{H}$. Thus, $\mathcal{H} = \{H_\alpha \mid \alpha < \omega\}$ for some ordinal number ω. Using the transfinite induction, let us now construct a collection of OAs, some $\mathbb{C} = \{L_\alpha \mid \alpha < \omega\}$ such that the following conditions are satisfied:

(i) If $\alpha \leq \beta < \omega$, then there is an OA-embedding $e_{\beta,\alpha}: L_\alpha \to L_\beta$ and, moreover, $e_{\gamma,\alpha} = e_{\gamma,\beta} \circ e_{\beta,\alpha}$ for any triple α, β, γ with $\alpha \leq \beta \leq \gamma < \omega$.

(ii) A state $s \in \mathcal{S}(L)$ admits a unique extension over L_α if and only if s fulfills all the inequalities of the set $\{H_\beta \mid \beta \leq \alpha\}$.

Indeed, if α is isolated, then we can directly apply Proposition 2.5. If α is a limit, then we first take the limit $\bar{L}$ of the collection $\{L_\beta \mid \beta < \alpha\}$ (Proposition 2.6) and then enlarge $\bar{L}$ to the desired L_α by a pasting that guarantees the inequality H_α (we use Proposition 2.5 again). In order to complete the proof, we now take the limit of the entire collection $\{L_\alpha \mid \alpha < \omega\}$. It follows from the construction that L will allow exactly the extensions of those states that satisfy all of the inequalities H_α ($\alpha < \omega$). This concludes the proof of Theorem 2.1. ■

As a corollary to Theorem 2.1 we obtain the following representation theorem. [It should be noted that this representation theorem may be understood as a generalization of Shultz's (1974) theorem.]

Theorem 2.7. Let S be a compact convex subset of a locally convex topological linear space. Then there is a proper OA L such that $\mathcal{S}(L)$ is affinely homeomorphic to S. Moreover, this affine homeomorphism maps Int S onto the set of all strictly positive states on L.

Proof. Without any loss of generality, S may be supposed a subset of $[0, 1]^I$ (Schaefer, 1971). The latter space is affinely homeomorphic to the state space of the OA which arises by the pasting of the collection $\{B_i \mid i \in I\}$, where each B_i is a four-element Boolean algebra, and where we assume that $B_i \cap B_j = \{0, 1\}$ for $i \neq j$.

By Theorem 2.1, there is a proper OA L such that $\mathcal{S}(L)$ is affinely homeomorphic to S. ■

3. EXTENSIONS OF STATES (THE CLASS *STEAD*)

In this section we shall be interested in extensions of states from suborthoalgebras over the entire orthoalgebras. A natural condition on OAs then is that they are unital [recall (Foulis and Bennett, 1993) that L is called *unital* if for any $a \in L$, $a \neq 0$, there exists a state s on L such that $s(a) = 1$]. (Indeed, by Theorem 2.1 every OA is a suborthoalgebra of a stateless OA!

It follows that the state extension problem would be meaningless for general OAs.)

In the sequel, the following class of orthoalgebras will play a central role in our investigations.

Definition 3.1. Let K be a unital OA. We say that K is *state-extension-admissible* if the following condition is satisfied: If K is a suborthoalgebra of a unital orthoalgebra L and if $s \in \mathcal{S}(K)$, then there is a state $t \in \mathcal{S}(L)$ such that $t \mid K = s$.

Thus, K is state-extension-admissible if all the states of K admit extensions over any unital OA that contains K. Let us denote the class of all state-extension-admissible OAs by *STEAD*. We shall show in this section that the class *STEAD* is (perhaps surprisingly) quite large and that it contains OAs familiar within quantum theories.

Let us first introduce a special type of state which will be relevant to our consideration here and may be found useful elsewhere, too.

Definition 3.2. Let K be an OA and let $s \in S(K)$. Then s is called *hyperpure* if there is an element $a \in K$ such that $s(a) = 1$ and, moreover, the state s is the only state with the latter property.

Let us denote by $\mathcal{S}_{\mathrm{hp}}(K)$, resp. $\mathcal{S}_{\mathrm{p}}(K)$, the set of all hyperpure, resp. pure, states on K. (A state is called *pure* if it cannot be written as a convex combination of two distinct states.) Obviously, every hyperpure state has to be pure. Thus, $\mathcal{S}_{\mathrm{hp}}(K) \subset \mathcal{S}_{\mathrm{p}}(K)$ for any K. One sees easily that if, for instance, K is a finite Boolean OA or if $K = L(H)$ for a Hilbert space H with dim $H < \infty$, then $\mathcal{S}_{\mathrm{hp}}(K) = \mathcal{S}_{\mathrm{p}}(K)$. According to Navara and Pták (1983) and Navara and Rogalewicz (1988), there are finite OAs (even orthomodular lattices) which possess pure states that are not hyperpure.

Our first result indicates the significance of hyperpure states in our context. [Recall that if $S \subset \mathcal{S}(K)$, then the symbol $\overline{conv}\ S$ denotes the topological closure in $\mathcal{S}(K)$ of the convex hull, *conv* S, of S.]

Theorem 3.3. Let K be a unital OA. If $\mathcal{S}_{\mathrm{p}}(K) \subset \overline{conv}\ \mathcal{S}_{\mathrm{hp}}(K)$, then $K \in$ *STEAD*.

Proof. Suppose that K is a suborthoalgebra of L and suppose further that L is unital. By the Krein–Millman theorem (see, e.g., Schaefer, 1971), $\overline{conv}\ \mathcal{S}_{\mathrm{p}}(K) = \mathcal{S}(K)$ and so $\overline{conv}\ \mathcal{S}_{\mathrm{hp}}(K) = \mathcal{S}(K)$. This implies that if every hyperpure state on K admits an extension over L, then so does every state on K. Indeed, if $s = \Sigma_{i\leq n} \alpha_i s_i$, where every s_i admits an extension over L, then so does s. Moreover, suppose that s_α ($\alpha \in I$) is a net in $\mathcal{S}(K)$ that converges to a $t \in \mathcal{S}(K)$. Suppose that $\tilde{s}_\alpha$ ($\alpha \in I$) is a net in $\mathcal{S}(L)$ of extensions of s_α. Since $\mathcal{S}(L)$ is compact, we obtain that a subset of $\tilde{s}_\alpha$ converges to a state $\tilde{t} \in \mathcal{S}(L)$. Since the convergence in $\mathcal{S}(K)$ is pointwise, we infer that $\tilde{t}|K = t$, which we wanted to check. What remains to show is that every hyperpure state of K admits an extension over L. To do this, let $s \in \mathcal{S}_{\mathrm{hp}}(K)$. Then there is an element $a \in K$ such that $s(a) = 1$ and, moreover, s is the only state of $\mathcal{S}(K)$ with the latter property. Since L is unital, there is a state $t \in \mathcal{S}(L)$ such that $t(a) = 1$. Put $u = t|K$. Since u is a state on K with $u(a) = 1$, we see that $u = s$. Thus, $s = t|K$ and this completes the proof of Theorem 3.3. ∎

The next result says that the sufficient condition for an OA to belong to *STEAD* which we have employed in Theorem 3.3 becomes also necessary if the state space of the OA in question is not too complex.

Theorem 3.4. Let K be a unital OA and let $\dim \mathcal{S}(K) < +\infty$. Then $K \in$ *STEAD* if and only if $\mathcal{S}_{\mathrm{p}}(K) \subset \overline{conv}\, \mathcal{S}_{\mathrm{hp}}(K)$.

Proof. We have to prove the necessity; the sufficiency follows from Theorem 3.3. We shall need a few lemmas. [The orthoalgebra K referred to in these lemmas is assumed to be unital and to satisfy the condition of $\dim \mathcal{S}(K) < \infty$.]

Lemma 3.5. Let $\mathcal{S}_{\mathrm{hp}}(K)$ be a unital set of states on K. Let $s \in \mathcal{S}(K)\backslash\overline{conv}\, \mathcal{S}_{\mathrm{hp}}(K)$ and let aff $\mathcal{S}(K)$ denote the affine hull of $\mathcal{S}(K)$ in the corresponding finite-dimensional Euclidean space. Then there is a closed half-space H of aff $\mathcal{S}(K)$ such that the following conditions are satisfied:

(i) $s \in \mathcal{S}(K)\backslash H$.

(ii) $\overline{conv}\, \mathcal{S}_{\mathrm{hp}}(K) \subset \mathrm{Int}\, H$.

(iii) There exist natural numbers $n, n_1, \ldots, n_m \in N$ and atoms $a_1, a_2, \ldots, a_m$ in K such that $n > n_i$ $(i \leq m)$, $\Sigma_{i\leq m} n_i \geq 3$, and such that $H = \{t \in \mathrm{aff}\, \mathcal{S}(K) \mid \Sigma_{i\leq m} n_i t(a_i) \leq n\}$.

Proof of Lemma 3.5. Observe first that we can easily find a closed half-space H of aff $\mathcal{S}(K)$ satisfying the conditions (i) and (ii). It suffices to put $H = \{t \in \mathrm{aff}\, \mathcal{S}(K) \mid f(t) \leq 0\}$, where f is a functional of the type $f(t) = \Sigma_{j\leq k} q_j t(c_j) - q$, $c_1, \ldots, c_k \in K$, $q, q_1, \ldots, q_k \in R\backslash\{0\}$. Since there is a nonempty open set of functionals with the properties (i) and (ii), we may choose f (and H) such that the coefficients $q, q_1, \ldots, q_k$ are rational. By multiplying f with a suitable integer, we may (and shall) suppose that $q, q_1, \ldots, q_k$ are (nonzero) integers.

For $j = 1, \ldots, k$ let us define

$$b_j = \begin{cases} c_j & \text{if } q_j > 0 \\ c_j' & \text{if } q_j < 0 \end{cases}$$

Then $f(t) = \Sigma_{j\leq k} |q_j| t(b_j) - n$, where $n = q + \Sigma_{j\leq k, q_j<0} q_j$.

Since K is unital, no mutually orthogonal subset of K has a cardinality larger than the affine dimension of $\mathcal{S}(K)$. This dimension is assumed to be finite, so each element of K can be expressed as a join of finitely many mutually orthogonal atoms. We can therefore write f in the form

$$f(t) = \sum_{i \leq m} n_i t(a_i) - n$$

where $a_1, \ldots, a_m$ are atoms of K and $\{n_i \mid i \leq m\} = \{|q_j| \mid j \leq k\} \subset N$. This means that H allows the expression of the form (iii). (If the condition $\Sigma_{i \leq m} n_i \geq 3$ is not satisfied, we triple all the coefficients $n, n_0, \ldots, n_m$.) The only condition which has to be verified is the inequality $n > n_i$ $(i \leq m)$. To this end, take for any a_i $(i \leq m)$ a state $s_i \in \overline{conv}\, \mathcal{S}_{\text{hp}}(K)$ such that $s_i(a_i) = 1$ (the existence of such a state is assumed in Lemma 3.5). It follows that $f(s_i) < 0$. We obtain

$$n_i = n_i s_i(a_i) \leq \sum_{j \leq m} n_j s_i(a_j) = f(s_i) + n < n$$

The proof of Lemma 3.5 is complete. ■

Lemma 3.6. Let $\overline{conv}\, \mathcal{S}_{\text{hp}}(K)$ be a unital set of states on K. Let $s \in \mathcal{S}(K) \backslash \overline{conv}\, \mathcal{S}_{\text{hp}}(K)$ and let H denote the closed half-space with the properties of Lemma 3.5. Then there is a unital OA L such that the following statement holds true: A state $t \in \mathcal{S}(K)$ admits an extension over L if and only if $t \in H$.

Proof. In view of Lemma 3.5, we have to show that there is an OA L containing K such that $t \in \mathcal{S}(K)$ admits an extension over L exactly if $\Sigma_{i \leq m} n_i t(a_i) \leq n$. Since $\overline{conv}\, \mathcal{S}_{\text{hp}}(K) \subset \operatorname{Int} H$, each $t \in \overline{conv}\, \mathcal{S}_{\text{hp}}(K)$ satisfies the inequality $\Sigma_{i \leq m} n_i t(a_i) < n$. Since $\overline{conv}\, \mathcal{S}_{\text{hp}}(K)$ is a compact set, there is an $\epsilon > 0$ such that $\Sigma_{i \leq m} n_i t(a_i) \leq n - \epsilon$. Multiplying all n, n_i $(i \leq m)$ by some (sufficiently large) integer, the inequality defining H remains unchanged. Thus, without any loss of generality, we shall suppose that the inequality

$$\text{(C)} \qquad \sum_{i \leq m} n_i t(a_i) \leq n - 1$$

is satisfied for all $t \in \overline{conv}\, \mathcal{S}_{\text{hp}}(K)$.

Notice that the number $p = \Sigma_{i \leq m} n_i - n$ is positive, for $s \notin H$ implies $n < \Sigma_{i \leq m} n_i s(a_i) \leq \Sigma_{i \leq m} n_i$. By Lemma 3.5 and Theorem 1.12, the Greechie diagram of Fig. 6 represents an OA. Let us denote the latter OA by $\tilde{K}$. Further, let us take such copies of K and $\tilde{K}$ that $K \cap \tilde{K}$ contains exactly $0, a_1, \ldots, a_m$ and the complements of the latter elements. We now take for L the pasting of K and $\tilde{K}$.

Obviously, K is a suborthoalgebra of L. Suppose that t is a state on K that admits an extension over L. Consider the "rectangular" parts of L (see

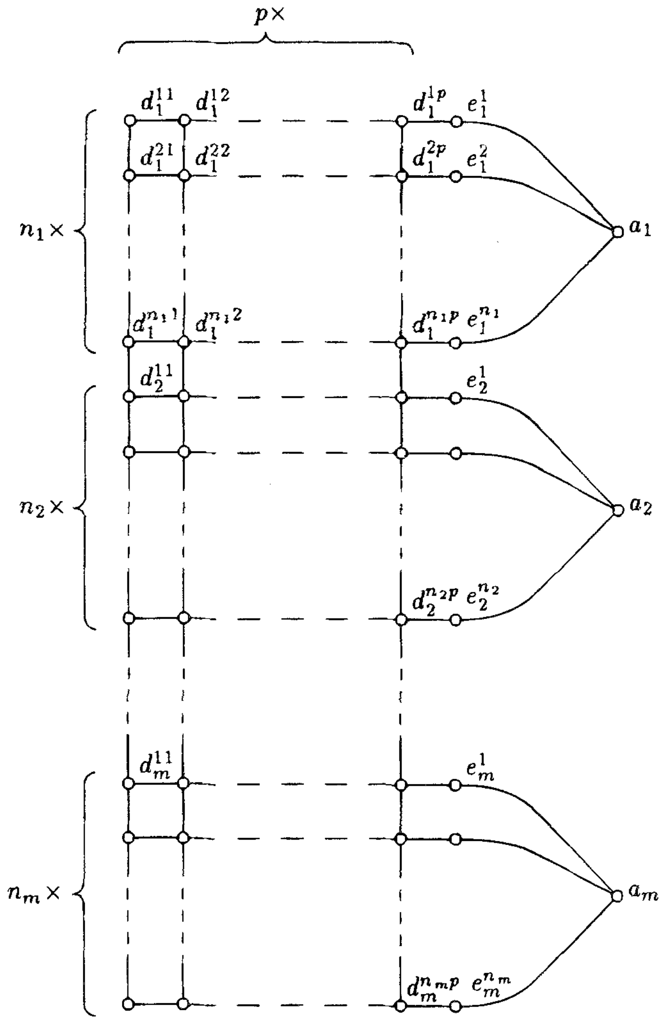

Fig. 6.

Fig. 6; there are m parts of this type there) that are covered by both "horizontal" and "vertical" blocks. The sum of values attained by t in any "row" is at most $1 - t(a_i)$ ($i \leq m$). The sum of values attained by t in any "column" is 1. It follows that $\Sigma_{i \leq m} n_i(1 - t(a_i)) \geq p$, which is equivalent to the required inequality $\Sigma_{i \leq m} n_i t(a_i) - n \leq 0$.

Let us on the contrary assume that $t \in \mathcal{S}(K)$ and that t satisfies the above inequality. We shall show that t admits an extension over L. Let us determine the values of a state $u \in \mathcal{S}(L)$ that will eventually become the required extension of t. [Thus we require $u(a_i) = t(a_i)$ for $i \leq m$.] Let us put $u(d_i^{jk}) = v_i$ for all i, j, k, where v_i are suitably chosen nonnegative numbers whose values will be specified later. We have $u(e_i^j) = 1 - t(a_i) - pv_i$. Since all $u(e_i^j)$ have to be nonnegative, we see that v_i have to satisfy the inequality $v_i \leq (1 - t(a_i))/p$. The columns yield the equality $\Sigma_{i\leq m} n_i v_i = 1$, and there are no other requirements on v_i $(i \leq m)$. It follows that if $\Sigma_{i\leq m} n_i v_i = 1$ and if, moreover, the inequalities $v_i \leq (1 - t(a_i))/p$ $(i \leq m)$ are satisfied, then u becomes a state which is an extension of t. But the latter two properties of v_i $(i \leq m)$ are easy to be fulfilled for we have $\Sigma_{i\leq m} n_i(1 - t(a_i))/p \geq 1$. We conclude that $u \in \mathcal{S}(L)$ is an extension of t.

Finally, we have to check that L is unital. Suppose that $c \in L$ with $c \neq 0$ and look for a state u on L satisfying $u(c) = 1$. The case $c \in K$ has been clarified above. Let $c \in L\backslash K$. We may restrict our attention to the case when c is an atom. We have to distinguish two cases:

1. Suppose that c is one of the atoms d_i^{jk}. We will now outline how one finds a state $u \in \mathcal{S}(L)$ such that $u(d_1^{11}) = 1$. (The remaining cases of d_i^{jk} can be argued similarly.)

There is a state $t \in \overline{conv}\, \mathcal{S}_{\mathrm{hp}}(K)$ such that $t(a_1') = 1$. In constructing its extension, some $u \in \mathcal{S}(L)$, put $u(d_1^{11}) = 1$ and further put $u(x) = 0$ for all atoms x of $L\backslash K$ orthogonal to d_1^{11}. For the atoms of $L\backslash K$ nonorthogonal to d_1^{11} we define $u(d_i^{jk}) = v_i$, $u(e_i^j) = 1 - t(a_i) - (p - 1)v_i$, where $v_i \geq 0$ $(i \leq m)$ are suitable constants. In order to have $u(e_i^j) \geq 0$, we require $v_i \leq (1 - t(a_i))/(p - 1)$, while the vertical blocks yield $(n_1 - 1)v_1 + \Sigma_{2\leq i\leq m} n_i v_i = 1$. Obviously, the constants v_i $(i \leq m)$ can be found provided

$$(n_1 - 1)(1 - t(a_1)) + \sum_{2\leq i\leq m} n_i(1 - t(a_i)) \geq p - 1$$

Since $t(a_1) = 0$, we may rewrite the latter condition in the form $\Sigma_{i\leq m} n_i(1 - t(a_i)) \geq p$. This inequality is equivalent to the inequality determining H and hence it is satisfied by t.

2. Suppose now that c is one of the atoms e_i^j. We will now outline the procedure for finding $u \in \mathcal{S}(L)$ for $c = e_1^1$ (the other cases argue similarly). Again, we take a state $t \in \overline{conv}\, \mathcal{S}_{\mathrm{hp}}(K)$ such that $t(a_1') = 1$, and we extend t to a state u on L. We use the procedure described above—for the atoms of $L\backslash K$ nonorthogonal to e_1^1, we define $u(d_i^{jk}) = v_i$, $u(e_i^j) = 1 - t(a_i) - pv_i$, where v_i are suitably chosen constants $(v_i \geq 0, i \leq m)$. We have to fulfil the inequalities $v_i \leq (1 - t(a_i))/p$ $(i \leq m)$ and also the equality $(n_1 - 1)v_1 + \Sigma_{2\leq i\leq m} n_i v_i = 1$. This means the inequality

$$(n_1 - 1)(1 - t(a_1)) + \sum_{2\leq i\leq m} n_i(1 - t(a_i)) \geq p$$

After substituting $t(a_1) = 0$, we obtain $\Sigma_{i\leq m} n_i(1 - t(a_i)) \geq p + 1$, which is equivalent to (C).

We have verified that L is unital and this completes the proof of Lemma 3.6. ■

Remark 3.7. The proof of Lemma 3.6 somewhat resembles the proof of Proposition 2.5. However, here we have required the unitality of the enlargement, while in Proposition 2.5 we have required the uniqueness of the extensions of states. These two requirements contradict each other and therefore essentially different constructions are needed.

Let us return to the proof of Theorem 3.4. Let $K \in STEAD$. Suppose that there is a state $s \in \mathcal{S}(K)\setminus\overline{conv}\ \mathcal{S}_{\mathrm{hp}}(K)$ and look for a contradiction. If K is embedded into a unital logic, all hyperpure states must have extensions. Therefore all states from $\overline{conv}\ \mathcal{S}_{\mathrm{hp}}(K)$ must have extensions, too. We see that $\overline{conv}\ \mathcal{S}_{\mathrm{hp}}(K)$ is a unital set of states on K. We now apply Lemma 3.6 to construct a unital enlargement L of K such that s cannot be extended to a state on L. This contradicts the assumption $K \in STEAD$ and the proof is complete. ■

Let us now formulate two explicit corollaries of the latter result.

Corollary 3.8. Let K be a finite unital OA. Then $K \in STEAD$ if and only if $\mathcal{S}_{\mathrm{p}}(K) = \mathcal{S}_{\mathrm{hp}}(K)$.

Proof. This follows directly from Theorem 3.4 and from the observation that if K is finite, then $\mathcal{S}(K)$ is a polytope. ■

Corollary 3.9. Let $K = L(H)$, where H is a finite-dimensional Hilbert space with dim $H \geq 3$. Then $K \in STEAD$.

Proof. This follows from Theorem 3.4 and Gleason's (1957) theorem. ■

In connection with the latter corollaries, let us make two illustrating remarks. First, there is a finite unital orthoalgebra K such that $\mathcal{S}_{\mathrm{p}}(K) \neq \mathcal{S}_{\mathrm{hp}}(K)$ (see, e.g., Navara and Pták, 1983). Thus, the class *STEAD* does not contain all finite unital OAs. On the other hand, there is a finite OA (indeed, an orthomodular lattice) belonging to the class *STEAD* which is not a Boolean algebra (Navara, 1987; Navara and Rogalewicz, 1988). Second, it should be noted that even in the case of $\mathcal{S}(K)$ being finite dimensional, the inclusion $\mathcal{S}_{\mathrm{p}}(K) \subset \overline{conv}\ \mathcal{S}_{\mathrm{hp}}(K)$ does not imply $\mathcal{S}_{\mathrm{p}}(K) = \mathcal{S}_{\mathrm{hp}}(K)$. Ovchinnikov (1993) presents an example of a countable OA (indeed, of an orthomodular lattice) K that is a suborthoalgebra of $L(R^3)$ and enjoys the following property: Every state on K admits a unique extension over $L(R^3)$. It follows that $\mathcal{S}_{\mathrm{p}}(K) \neq \mathcal{S}_{\mathrm{hp}}(K)$, although $\mathcal{S}(K) = \overline{conv}\ \mathcal{S}_{\mathrm{hp}}(K)$ (and therefore $K \in STEAD$).

In view of the distinguished position of Hilbertian logics in quantum axiomatics, a natural question arises of whether Corollary 3.9 can be extended to arbitrary Hilbert spaces. The following result says that this is indeed so.

Theorem 3.10. Let $K = L(H)$, where H is a complex Hilbert space and $\dim H \neq 2$. Then $K \in STEAD$.

Proof. Let us first prove the following auxiliary result.

Lemma 3.11. Let H be a complex Hilbert space and let x be a unit vector of H. Let s_x denote the state on $L(H)$ determined by the formula $s_x(P) = \|P(x)\|^2$, where $P \in L(H)$. Then $s_x \in \mathcal{S}_{\mathrm{hp}}(L(H))$.

Proof of Lemma 3.11. Let P_x denote the orthogonal projection onto the linear span of x in H and let s be a state in $\mathcal{S}(L(H))$ such that $s(P_x) = 1$. By the theorem of Aarnes (1970), we can write $s = s_1 + s_2$, where s_1 is a completely additive measure on $L(H)$ and s_2 is a measure on $L(H)$ which vanishes on every finite-dimensional projection. We see therefore that $s_2(P_x^\perp) \leq s(P_x^\perp) = 0$ and also $s_2(P_x) = 0$. This means that $s_2(I) = s_2(P_x \cup P_x^\perp) = s_2(P_x) + s_2(P_x^\perp) = 0$ (I is the identity operator here) and this implies that $s_2(P) = 0$ for any $P \in L(H)$. We infer that $s = s_1$ and therefore s is completely additive. By a generalized version of Gleason's theorem (Bunce and Wright, 1992), we see that $s = \sum_{n=1}^\infty \alpha_n t_n$, where the states t_n ($n \in N$) live on mutually orthogonal one-dimensional projections of H. Since $s(P_x) = 1$ and since $\sum_{n=1}^\infty \alpha_n = 1$, we conclude that for all but one $n \in N$ we have $\alpha_n = 0$. There is a single $n_0 \in N$ such that $\alpha_{n_0} = 1$ and it follows that $t_{n_0} = s_x$. This completes the proof of Lemma 3.11. ■

Let us return to the proof of Theorem 3.10. We shall apply Bunce and Wright (1992) again: The states on $L(H)$ are in a "true" one-to-one correspondence with the "functional" states on the von Neumann algebra $\mathcal{B}(H)$ of all bounded linear operators on H. (Here we have used the word true to indicate that the correspondence preserves the natural algebraic and topological properties.) In particular, the pure states on $L(H)$, $\mathcal{S}_{\mathrm{p}}(L(H))$ are in a one-to-one correspondence with pure states on $\mathcal{B}(H)$, $\mathcal{S}_{\mathrm{p}}(\mathcal{B}(H))$, and the hyperpure states on $L(H)$, $\mathcal{S}_{\mathrm{hp}}(L(H))$ are in a one-to-one correspondence with the functional hyperpure states on $\mathcal{B}(H)$, $\mathcal{S}_{\mathrm{hp}}(\mathcal{B}(H))$. Since $A \in \mathcal{B}(H)$ is nonnegative if and only if $s_x(A) \geq 0$ for all $x \in H$ ($\|x\| = 1$), we have

$$\mathcal{S}(\mathcal{B}(H)) = \overline{\mathrm{conv}}\{s_x \mid x \in H, \|x\| = 1\} = \overline{\mathrm{conv}}\, \mathcal{S}_{\mathrm{hp}}(\mathcal{B}(H))$$

[see, e.g., Kadison and Ringrose (1986), Theorem 4.3.9, p. 262]. It follows that $\mathcal{S}_{\mathrm{p}}(L(H)) \subset \overline{\mathrm{conv}}\, \mathcal{S}_{\mathrm{hp}}(L(H))$ and the proof is finished by applying Theorem 3.3. This completes the proof of Theorem 3.10. ■

Let us finally take up another basic case of OAs—the case of K being a Boolean OA. Obviously, if K is atomistic, then $K \in STEAD$ by Theorem 3.3. [Indeed, if K is Boolean, then $\mathcal{S}_{\mathrm{p}}(K)$ consists of two-valued states; see, e.g., Pták and Pulmannová (1991). If K is atomistic, we immediately obtain that $\mathcal{S}_{\mathrm{p}}(K) \subset \overline{conv}\ \mathcal{S}_{\mathrm{hp}}(K)$.] However, the following fully general result is in force here.

Theorem 3.12. Let K be a Boolean OA. Then $K \in STEAD$.

Proof. Assume that K is Boolean and assume further that K is a suborthoalgebra of a unital orthoalgebra L. Put $S = \{s \in \mathcal{S}(K) \mid s$ admits an extension over $L\}$. We want to show that $S = \mathcal{S}(K)$. It suffices to prove that S is both closed and dense in $\mathcal{S}(K)$.

Obviously, S is closed in $\mathcal{S}(K)$. Indeed, if $s_\alpha \in \mathcal{S}(K)$ is a net in S that converges to s and if $t_\alpha \in \mathcal{S}(L)$ is an extension of s_α, then the compactness of $\mathcal{S}(L)$ ensures that a subnet of t_α converges to a state. If $t \in \mathcal{S}(L)$ is this state, it is evident that t extends s and therefore S is closed in $\mathcal{S}(K)$.

To show that S is dense in $\mathcal{S}(K)$, let us assume that $s \in \mathcal{S}(K)$. Let

$$\mathbb{O}^{\epsilon}_{a_1,a_2,\ldots,a_n} = \{t \in \mathcal{S}(K) \,\big|\, |s(a_i) - t(a_i)| < \epsilon, i = 1, \ldots, n\}$$

where $\epsilon > 0$ and $a_1, a_2, \ldots, a_n \in K$ [thus $\mathbb{O}^{\epsilon}_{a_1,a_2,\ldots,a_n}$ is a standard neighborhood of s in $\mathcal{S}(K)$]. We shall prove that there is a state $\tilde{s}$ in $\mathbb{O}^{\epsilon}_{a_1,a_2,\ldots,a_n}$ which belongs to S. One obtains easily from the definition of the suborthoalgebra that the set $\{a_1, a_2, \ldots, a_n\}$ generates a finite suborthoalgebra in L that is Boolean. Let us denote the latter finite Boolean suborthoalgebra of L by B and let $\{b_1, b_2, \ldots, b_n\}$ be its atoms. Since L is unital, there are states $t_i \in \mathcal{S}(L)$ $(i \leq m)$ such that $t_i(b_i) = 1$. Obviously, $\Sigma_{i \leq m}\, s(b_i) = 1$ and therefore $t = \Sigma_{i \leq m}\, s(b_i)t_i$ is a state on L. Moreover, a simple computation yields that $t(a_i) = s(a_i)$ for any i $(i \leq m)$. Thus, t restricted to K is a state on K which is near to s within $\mathbb{O}^{\epsilon}_{a_1,a_2,\ldots,a_n}$. By the construction, $t \in S$. We have thus shown that S is dense in $\mathcal{S}(K)$ and this completes the proof of Theorem 3.12. ∎

Let us note in concluding that Theorem 3.3 and a combination of the reasoning of the proofs of Theorems 3.10 and 3.12 allow us to extend the latter results to products of Boolean and Hilbertian OAs. Since these "coupled structures" may be of importance within quantum theories (see, e.g., Aerts, 1981; Foulis and Pták, 1994; Pták and Pulmannová, 1991), let us conclude the paper by formulating this result.

Theorem 3.13. Let K_1 be a Boolean OA and let K_2 be a Hilbertian OA. Then $K_1 \times K_2 \in STEAD$.

ACKNOWLEDGMENT

The authors gratefully acknowledge the support of grant 201/93/0953 of the Grant Agency of the Czech Republic.

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